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Procedia IUTAM 11 (2014) 69 – 80
Procedia IUTAM
www.elsevier.com/locate/procedia

Nonlinear Interfacial Wave Phenomena from the Micro- to the Macro-Scale

Analyticity for Kuramoto–Sivashinsky type equations and related systems

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Abstract

We study the analyticity properties of solutions of Kuramoto–Sivashinsky type equations and related systems, with periodic initial data. In order to do this, we explore the sharpness of the method developed in Collet *et al.*[5], by investigating its applicability to other models. We prove that the solutions of a variety of dissipative-dispersive systems, which possess a global attractor, are analytic with respect to the spatial variable in a strip around the real axis; and a lower bound for the width of the strip of analyticity is obtained in each case.

Selection and peer-review under responsibility of scientific committee of Nonlinear Interfacial Wave Phenomena from the Micro- to the Macro-Scale

Keywords: Kuramoto–Sivashinsky equation, dissipative-dispersive evolutionary equations, dispersive pseudo-differential operators, analyticity of solutions of partial differential equations.

1. Physical models

1.1. The Kuramoto–Sivashinsky equation

The Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{1.1}$$

is a simple nonlinear PDE which exhibits complex spatio-temporal dynamics. It has been derived in the context of plasma ion mode instabilities by LaQuey *et al.*[16], reaction-diffusion systems by Kuramoto and Tsuzuki[15], laminar flame fronts by Sivashinsky[21] and viscous liquid flows on an inclined plane by Sivashinsky and Michelson[22].

The L-periodic in space solutions of (1.1), i.e., $u(x+L,t) = u(x,t)$, have received considerable attention both analytically and computationally. Its conservative nature allows us to restrict our attention to zero average solutions of (1.1). If we express a solution u of (1.1) as a Fourier series $u(x,t) = \sum_{\mu\in q\mathbb{Z}} \hat{u}(\mu,t)e^{i\mu x}$, where $q = 2\pi/L$, then its Fourier

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Selection and peer-review under responsibility of scientific committee of Nonlinear Interfacial Wave Phenomena from the Micro- to the Macro-Scale
doi:10.1016/j.piutam.2014.01.049

coefficients satisfy the infinite dimensional dynamical system

$$\frac{d}{dt}\hat{u}(\mu,t) = (\mu^2-\mu^4)\,\hat{u}(\mu,t) - \frac{i\mu}{2}\sum_{\mu'\in q\mathbb{Z}}\hat{u}(\mu',t)\,\hat{u}(\mu-\mu',t), \quad \mu\in q\mathbb{Z}. \tag{1.2}$$

Equations (1.2) reveal that *high* frequencies ($|\mu|>1$) are linearly stable, while the *low* frequencies ($0<|\mu|<1$) are linearly unstable. The nonlinear term in (1.2) causes transfer of energy from low to high frequencies and keeps the solution of (1.1) bounded in the L^2–norm.

It is shown independently by Il'yashenko[11], Goodman[10] and Collet *et al.*[4] that the solutions of (1.1) are bounded, for any L-periodic initial data. In[6] it is also shown that bounds in the L^2-norm of the solution of KS equation imply that these solutions are attracted by a set of finite dimension, the global attractor, and bounds in the dimension of the global attractor are provided.

Collet *et al.*[5] using a semigroup method established the analyticity of solutions of KS equation. In particular, they show that at large times the solution is analytic in a strip of size $\gamma_L \geq c\,L^{-16/25}$ around the real axis, where c is a constant independent of L. This provides the following estimate for the spectral density at high wavenumbers,

$$\limsup_{t\to\infty}|\hat{u}(j,t)| = O(e^{-cL^{-16/25}q|j|}),$$

where $\hat{u}(j,t)$ is the jth Fourier coefficient of $u(\cdot,t)$ and $q=\frac{2\pi}{L}$.

1.2. The dispersively modified Kuramoto–Sivashinsky equation

The equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u = 0, \tag{1.3}$$

where $\nu>0$ and $\mathcal{D}$ is a dispersive pseudo-differential operator (i.e., $\operatorname{Re}\widehat{\mathcal{D}}(i\xi)=0$, where ξ is the wave number in Fourier space), has been derived in the context of interfacial hydrodynamics. For example, Papageorgiou *et al.*[20] and Kas-Danouche *et al.*[13] derived this equation to describe the stability of core-annular flows with applications to oil transport (lubricated pipe-lining). In the latter derivation $\widehat{\mathcal{D}}$ can be expressed as

$$\widehat{\mathcal{D}u}(\xi) = \frac{i\xi^2 I_1(\xi)}{\xi I_1^2(\xi) - \xi I_0^2(\xi) + 2I_0(\xi)I_1(\xi)}\hat{u}(\xi),$$

where $I_n(\xi)$ denotes the modified Bessel function of the first kind of order n. The well-posedness of (1.3) for periodic initial data can be derived from the work of Tadmor[23] since it constitutes a special case of the central theorem proved there. In particular, it can be shown that the corresponding initial value problem possesses a global (space periodic) solution which grows at most exponentially in time. For (one–dimensional) falling film flows a particular case of (1.3) where $\mathcal{D}u=\delta u_{xxx}$ and δ is a constant was originally derived by Topper and Kawahara[24]. The resulting equation is the following KS/KdV equation[1]

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + \nu u_{xxxx} = 0. \tag{1.4}$$

Kawahara and Toh[14] were among the first to establish the regularizing effect of dispersion on the dynamics with traveling wave pulses emerging at large times. It is noteworthy that equation (1.4) with the inclusion of a fifth order dispersion term

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + \nu u_{xxxx} + \varepsilon u_{xxxxx} = 0, \tag{1.5}$$

known as the Benney–Lin equation, has been derived in the context of one–dimensional evolutions of small but finite amplitude long waves in various problems in fluid dynamics. (See for example Benney[2] and Lin[17].) Global well–posedness of the initial value problem of (1.5) with initial data in $H^s(\mathbb{R})$, $s\geq 0$, has been established by Biagioni and Linares[3].

[1] Also known as Kawahara equation.

1.3. Nonlocal Kuramoto–Sivashinsky equations

The equation

$$u_t + uu_x \pm u_{xx} + \nu u_{xxxx} + \mu \mathcal{H}[u]_{xxx} = 0, \tag{1.6}$$

with $\nu > 0$, $\mu \geq 0$ and $\mathcal{H}$ the Hilbert transform operator defined by

$$\mathcal{H}[f](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi} d\xi,$$

(in Fourier space $(\widehat{\mathcal{H}[w]})_k = -i\,\mathrm{sgn}(\mathrm{Re}k)\hat{w}_k$), exhibits a complex behavior including chaotic oscillations as in the case of the usual KS equation. This equation was first derived by Gonzales and Castellanos[9] and recently by Tseluiko and Papageorgiou[26] using formal asymptotics. A plus sign in front of the u_{xx} term corresponds to the linearly unstable hydrodynamic regime (the modified Kuramoto–Sivashinsky (MKS) equation) and a minus sign to the stable one (the modified damped Kuramoto–Sivashinsky (MDKS) equation). A weakly nonlinear analysis of the Navier–Stokes equations, the electrostatics equations and associated free surface conditions, leads to a MKS, or a MDKS equation which have an additional nonlocal term due to the effect of the electric field.

1.4. The Burgers–Sivashinsky equation

The Burgers–Sivashinsky (BS) equation

$$u_t + uu_x - u - u_{xx} = 0,$$

superficially seems to have much in common with the KS equation. It has low wave number instability, high wave number damping, and nonlinear stabilization via energy transfer. Despite the similarity between KS and BS, when L is large their solutions have different qualitative behavior. KS solutions are observed to have high dimensional chaos (see[18]) while BS solutions just approach time independent steady states as $t \to \infty$.

1.5. The $\boldsymbol{\alpha,\beta}$-model

Of interest is the model equation

$$u_t + uu_x - |\partial_x|^{\alpha} u + |\partial_x|^{\beta} u \;=\; 0, \tag{1.7}$$

with $L-$periodic initial data, where $\beta > \alpha \geq 0$ and the operator $|\partial_x|^{\sigma}$ is defined by

$$|\partial_x|^{\sigma}\Big(\sum_{k\in\mathbb{Z}} \hat{w}_k\, e^{iqkx}\Big) \;=\; \sum_{k\in\mathbb{Z}} q^{\sigma}|k|^{\sigma}\hat{w}_k\, e^{iqkx}, \quad q \;=\; \frac{2\pi}{L},$$

which is due to Otto[19].

2. Analyticity for a class of dispersive-dissipative equations

In this work, we investigate the applicability of the *semigroup method* developed in Collet *et al.*[5] to a variety dissipative-dispersive equations. More specifically, we present analyticity properties of zero mean, spatially $L-$periodic solutions of equations of the type

$$u_t + uu_x + \mathcal{P}u \;=\; 0, \tag{2.1}$$

possessing a global attractor. Here, $\mathcal{P}$ is a pseudo-differential operator defined in Fourier space by

$$(\widehat{\mathcal{P}w})_k = \lambda_k\, \hat{w}_k,\; k \in \mathbb{Z},$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k\, e^{iqkx}$, $q = 2\pi/k$, and with the eigenvalues λ_k satisfying

$$\mathrm{Re}\,\lambda_k \geq c_1|k|^{\gamma} \text{ for all } |k| \geq k_0,$$

for some positive constants c_1, γ and k_0 a sufficiently large positive integer. Global existence of solutions of (2.1) has been established for $\gamma > \frac{3}{2}$ (see[23]); when $\gamma \geq 2$, it can be deduced from[8] that equation (2.1) possesses a global attractor compact in every Sobolev norm. Analyticity of solutions of (2.1) is established when $\gamma > \frac{5}{2}$, in[1]. Recently, in[12] it has been proven the analyticity of solutions of (2.1) when $\gamma > 2$.

2.1. The dispersively modified Otto's model

The dispersively modified KS equation is a special case of the equation

$$u_t + uu_x - |\partial_x|^{\alpha} u + |\partial_x|^{\beta} u + \mathcal{D}u = 0, \tag{2.2}$$

for $L-$periodic initial data, with $\beta > \alpha \geq 0$ and $\mathcal{D}$ is a dispersive pseudo-differential operator. In fact, (2.2) reduces to the KS equation for $\alpha = 2, \beta = 4$ and $\mathcal{D} \equiv 0$.

We shall prove that the solutions of (2.2) are analytic in a strip around the real axis.

We shall be using the inner product $(v, w) = \int_0^L v(x)\overline{w}(x)\, dx$ and the norm $\|u\| = (u, u)^{1/2}$, for u, v square integrable $L-$periodic functions. Let A be the operator defined by

$$(Av)(x) = \sum_{k\in\mathbb{Z}} q|k|\hat{v}_k e^{iqkx}, \quad \text{for } v(x) = \sum_{k\in\mathbb{Z}} \hat{v}_k e^{ikx}. \tag{2.3}$$

The operator $e^{\theta tA}$, with $\theta, t \in \mathbb{R}$ is defined by

$$(e^{\theta tA} v)(x) = \sum_{k\in\mathbb{Z}} e^{\theta tq|k|}\hat{v}_k e^{iqkx}.$$

Let u be an L-periodic solution (2.2). Let $v(x, t) := (e^{\theta tA} u)(x, t)$, where $\theta > 0$. Then, (2.2) becomes

$$(e^{-\theta tA})v_t - \theta e^{-\theta tA} Av + e^{-\theta tA}(-|\partial_x|^{\alpha} v + |\partial_x|^{\beta} v) + uu_x + \mathcal{D}u = 0. \tag{2.4}$$

Taking in (2.4) the inner product with $e^{\theta tA} v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(\cdot, t)\|^2 - \theta(Av, v) - \||\partial_x|^{\frac{\alpha}{2}} v\|^2 + \||\partial_x|^{\frac{\beta}{2}} v\|^2 + (\mathcal{D}u, e^{\theta tA} v) + (uu_x, e^{\theta tA} v) = 0. \tag{2.5}$$

We next define the trilinear form

$$b(v_1, v_2, v_3) := \int_0^L v_1(x)(\partial_x v_2)(x) v_3(x)\, dx.$$

Clearly, $b(u, u, e^{\theta tA} v) = (uu_x, e^{\theta tA} v)$. We shall need the following result (for proof see[5]).

Lemma 2.1. *There exists a constant C such that*

$$|b(u, u, e^{\theta tA} v)| \leq C\sqrt{\theta t}\, \|v\|\, \|Av\|^2. \tag{2.6}$$

Combining (2.5) with (2.6), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \theta\|A^{\frac{1}{2}} v\|^2 + \|A^{\frac{\alpha}{2}} v\|^2 - \|A^{\frac{\beta}{2}} v\|^2 + C\sqrt{\theta t}\, \|v\|\, \|Av\|^2. \tag{2.7}$$

Lemma 2.2. *For every $\beta > \alpha \geq 0$,*

$$\|A^{\frac{\alpha}{2}} v\|^2 \leq \|v\|^{2-\frac{2\alpha}{\beta}}\, \|A^{\frac{\beta}{2}} v\|^{\frac{2\alpha}{\beta}}. \tag{2.8}$$

Proof. Using Hölder's inequality, we obtain

$$\begin{aligned}\|A^{\frac{\alpha}{2}}v\|^2 &= \sum_{k=1}^{\infty}|k|^{\alpha}|v_k|^2 = \sum_{k=1}^{\infty}|k|^{\alpha}|v_k|^{\frac{2\alpha}{\beta}}|v_k|^{\frac{2\beta-2\alpha}{\beta}}\\ &\le \left(\sum_{k=1}^{\infty}\left(|v_k|^{\frac{2(\beta-\alpha)}{\beta}}\right)^{\frac{\beta}{\beta-\alpha}}\right)^{\frac{\beta-\alpha}{\beta}}\left(\sum_{k=1}^{\infty}\left(|k|^{\alpha}|v_k|^{\frac{2\alpha}{\beta}}\right)^{\frac{\beta}{\alpha}}\right)^{\frac{\alpha}{\beta}}\\ &= \left(\sum_{k=1}^{\infty}|v_k|^2\right)^{\frac{\beta-\alpha}{\beta}}\left(\sum_{k=1}^{\infty}|k|^{\beta}|v_k|^2\right)^{\frac{\alpha}{\beta}} = \|v\|^{2-\frac{2\alpha}{\beta}}\,\|A^{\frac{\beta}{2}}v\|^{\frac{2\alpha}{\beta}}.\end{aligned}$$

□

Combining, (2.7) and (2.8), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \theta\|v\|^{2-\frac{2}{\beta}}\,\|A^{\frac{\beta}{2}}v\|^{\frac{2}{\beta}} + \|v\|^{2-\frac{2\alpha}{\beta}}\,\|A^{\frac{\beta}{2}}v\|^{\frac{2\alpha}{\beta}} - \|A^{\frac{\beta}{2}}v\|^2 + C\sqrt{\theta t}\,\|v\|^{3-\frac{4}{\beta}}\,\|A^{\frac{\beta}{2}}v\|^{\frac{4}{\beta}}.$$

Using here Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \le{}& \frac{\left(\frac{\theta}{\varepsilon_1}\|v\|^{2-\frac{2}{\beta}}\right)^{\frac{\beta}{\beta-1}}}{\frac{\beta}{\beta-1}} + \frac{\left(\varepsilon_1\|A^{\frac{\beta}{2}}v\|^{\frac{2}{\beta}}\right)^{\beta}}{\beta} + \frac{\left(\frac{1}{\varepsilon_2}\|v\|^{2-\frac{2\alpha}{\beta}}\right)^{\frac{\beta}{\beta-\alpha}}}{\frac{\beta}{\beta-\alpha}} + \frac{\left(\varepsilon_2\|A^{\frac{\beta}{2}}v\|^{\frac{2\alpha}{\beta}}\right)^{\frac{\beta}{\alpha}}}{\frac{\beta}{\alpha}}\\ &- \|A^{\frac{\beta}{2}}v\|^2 + \frac{\left(\frac{1}{\varepsilon_3}C\sqrt{\theta t}\|v\|^{3-\frac{4}{\beta}}\right)^{\frac{\beta}{\beta-2}}}{\frac{\beta}{\beta-2}} + \frac{\left(\varepsilon_3\|A^{\frac{\beta}{2}}v\|^{\frac{4}{\beta}}\right)^{\frac{\beta}{2}}}{\frac{\beta}{2}},\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \le{}& \frac{(\beta-1)\theta^{\frac{\beta}{\beta-1}}}{\beta\varepsilon_1^{\frac{\beta}{\beta-1}}}\|v\|^2 + \frac{\varepsilon_1^{\beta}\|A^{\frac{\beta}{2}}v\|^2}{\beta} + \frac{\beta-\alpha}{\beta\varepsilon_2^{\frac{\beta}{\beta-\alpha}}}\|v\|^2 + \frac{\alpha\varepsilon_2^{\frac{\beta}{\alpha}}\|A^{\frac{\beta}{2}}v\|^2}{\beta} - \|A^{\frac{\beta}{2}}v\|^2\\ &+ \frac{(\beta-2)C^{\frac{\beta}{\beta-2}}(\theta t)^{\frac{\beta}{2(\beta-2)}}}{\beta\varepsilon_3^{\frac{\beta}{\beta-2}}}\|v\|^{\frac{3\beta-4}{\beta-2}} + \frac{2\varepsilon_3^{\frac{\beta}{2}}\|A^{\frac{\beta}{2}}v\|^2}{\beta},\end{aligned}$$

whence, choosing $\varepsilon_1 = (\beta-2)^{\frac{1}{\beta}}$, $\varepsilon_2 = \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\beta}}$, $\varepsilon_3 = \left(\frac{1}{2}\right)^{\frac{2}{\beta}}$ we get

$$\frac{d}{dt}\|v\|^2 \le \left(\frac{2(\beta-1)\theta^{\frac{\beta}{\beta-1}}}{\beta(\beta-2)^{\frac{1}{\beta-1}}} + \frac{2(\beta-\alpha)}{\beta\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\beta-\alpha}}}\right)\|v\|^2 + \frac{2(\beta-2)C^{\frac{\beta}{\beta-2}}(\theta t)^{\frac{\beta}{2(\beta-2)}}}{\beta\left(\frac{1}{2}\right)^{\frac{2}{\beta-2}}}\|v\|^{\frac{3\beta-4}{\beta-2}}. \tag{2.9}$$

With $\Phi(t) := \|v(\cdot,t)\|^2$, we write (2.9) in the form

$$\Phi'(t) \le \left(C_1\theta^{\frac{\beta}{\beta-1}} + C_2\right)\Phi(t) + C_3(\theta t)^{\frac{\beta}{2(\beta-2)}}\left(\Phi(t)\right)^{\frac{3\beta-4}{2(\beta-2)}}. \tag{2.10}$$

Assume now that $\Phi(0) \le R_L^2$. As long as $\Phi(t) \le 4R_L^2$ holds, relation (2.10) implies

$$\Phi'(t) \le \left(C_1\theta^{\frac{\beta}{\beta-1}} + C_2 + C_3(2R_L)^{\frac{\beta}{\beta-2}}(\theta t)^{\frac{\beta}{2(\beta-2)}}\right)\Phi(t),$$

whence

$$\Phi(t) \le \Phi(0)\exp\left[\left(C_1\theta^{\frac{\beta}{\beta-1}} + C_2\right)t + \frac{2(\beta-2)C_3(2R_L)^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}}\right].$$

As long as

$$\left(C_1\theta^{\frac{\beta}{\beta-1}}+C_2\right)t+\frac{2(\beta-2)C_3(2R_L)^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}}\leq\log 4,$$

we obviously have $\Phi(t)\leq 4R_L^2$. This holds for $t\leq t_L$ which is the positive root of the equation

$$\frac{2(\beta-2)C_3(2R_L)^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}}+\left(C_1\theta^{\frac{\beta}{\beta-1}}+C_2\right)t-\log 4=0.$$

Let us explain why

$$\frac{2(\beta-2)C_3(2R_L)^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}}+\left(C_1\theta^{\frac{\beta}{\beta-1}}+C_2\right)t-\log 4\leq 0,\ \text{when}\ t\in[0,t_L]. \tag{2.11}$$

First, note that $\kappa:=\frac{3\beta-4}{2(\beta-2)}\in(\frac{3}{2},\infty)$ when $\beta>2$. Also we have that both

$$C_4:=\frac{2(\beta-2)C_3(2R_L)^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}\quad\text{and}\quad\mu:=C_1\theta^{\frac{\beta}{\beta-1}}+C_2\quad\text{are positive.}$$

Consider now the function

$$f(t)\ =\ C_4t^{\kappa}+\mu t-\log 4,$$

where $t\in[0,\infty)$, with $\kappa\in(\frac{3}{2},\infty)$, $C_4>0$ and $\mu>0$. We have that $f(0)=-\log 4<0$ and $\lim_{t\to\infty}f(t)=\infty$. For the derivative of f we have that

$$f'(t)\ =\ \kappa C_4t^{\kappa-1}+\mu,$$

hence $f'>0$ in $(0,\infty)$ and so the function f is strictly increasing in $(0,\infty)$. Since f is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(t_L,0)$; and (2.11) is clear. The objective is to maximize the product θt, for large R_L, in order to optimize the width of the strip of analyticity.

We want to maximize $H(\theta,t)=\theta t$, subject to

$$F(\theta,t)=C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}t^{\frac{3\beta-4}{2(\beta-2)}}+C_1\theta^{\frac{\beta}{\beta-1}}t+C_2t-\log 4\leq 0. \tag{2.12}$$

Note that, it suffices to maximize H subject to $F(\theta,t)=0$, instead of (2.12). We consider the Lagrange function

$$G(\theta,t,\lambda)=\theta t+C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}t^{\frac{3\beta-4}{2(\beta-2)}}\lambda+C_1\theta^{\frac{\beta}{\beta-1}}t\lambda+C_2t\lambda-(\log 4)\lambda.$$

For the partial derivatives of G we have that

$$\begin{aligned}
G_\theta(\theta,t,\lambda)&=\ t+\frac{\beta}{2(\beta-2)}C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{4-\beta}{2(\beta-2)}}t^{\frac{3\beta-4}{2(\beta-2)}}\lambda+\frac{\beta}{\beta-1}C_1\theta^{\frac{1}{\beta-1}}t\lambda,\\
G_t(\theta,t,\lambda)&=\ \theta+\frac{3\beta-4}{2(\beta-2)}C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}t^{\frac{\beta}{2(\beta-2)}}\lambda+C_1\theta^{\frac{\beta}{\beta-1}}\lambda+C_2\lambda,\\
G_\lambda(\theta,t,\lambda)&=\ C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}t^{\frac{3\beta-4}{2(\beta-2)}}+C_1\theta^{\frac{\beta}{\beta-1}}t+C_2t-\log 4.
\end{aligned}$$

Now, we have to solve the system $G_\theta=G_t=G_\lambda=0$. More precisely we have the system

$$t+\frac{\beta}{2(\beta-2)}C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{4-\beta}{2(\beta-2)}}t^{\frac{3\beta-4}{2(\beta-2)}}\lambda+\frac{\beta}{\beta-1}C_1\theta^{\frac{1}{\beta-1}}t\lambda=0, \tag{2.13}$$

$$\theta+\frac{3\beta-4}{2(\beta-2)}C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}t^{\frac{\beta}{2(\beta-2)}}\lambda+C_1\theta^{\frac{\beta}{\beta-1}}\lambda+C_2\lambda=0, \tag{2.14}$$

$$C_5R_L^{\frac{\beta}{\beta-2}}\theta^{\frac{\beta}{2(\beta-2)}}t^{\frac{3\beta-4}{2(\beta-2)}}+C_1\theta^{\frac{\beta}{\beta-1}}t+C_2t-\log 4=0. \tag{2.15}$$

Multiplying (2.13) with θ and (2.14) with $-t$, and then summing the two equations we get

$$-C_5 R_L^{\frac{\beta}{\beta-2}} \theta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} + \frac{1}{\beta-1} C_1 \theta^{\frac{\beta}{\beta-1}} t - C_2 t = 0. \tag{2.16}$$

Summing (2.15) and (2.16), we get

$$t = \frac{(\beta-1)\log 4}{\beta C_1} \theta^{-\frac{\beta}{\beta-1}}, \text{ i.e., } t = C_6 \theta^{-\frac{\beta}{\beta-1}}. \tag{2.17}$$

Combining (2.15) and (2.17), we obtain that

$$C_7 R_L^{\frac{\beta}{\beta-2}} \theta^{-\frac{\beta(2\beta-3)}{2(\beta-1)(\beta-2)}} + C_2 C_6 \theta^{-\frac{\beta}{\beta-1}} + C_1 C_6 - \log 4 = 0. \tag{2.18}$$

Now, note that $C_1 C_6 - \log 4 = -\frac{\log 4}{\beta} < 0$. Let $\xi = \theta^{-\frac{\beta}{2(\beta-1)(\beta-2)}}$ and (2.18) becomes

$$\xi^{2\beta-3} + \frac{C_8}{R_L^{\frac{\beta}{\beta-2}}} \xi^{2(\beta-2)} - \frac{C_9}{R_L^{\frac{\beta}{\beta-2}}} = 0.$$

Consider now the function

$$g(\xi) = \xi^{2\beta-3} + \frac{C_8}{R_L^{\frac{\beta}{\beta-2}}} \xi^{2(\beta-2)} - \frac{C_9}{R_L^{\frac{\beta}{\beta-2}}}, \quad \text{where } \xi \in [0, \infty).$$

We have that $g(0) = -\frac{C_9}{R_L^{\frac{\beta}{\beta-2}}} < 0$ and $\lim_{\xi\to\infty} g(\xi) = \infty$. For the derivative of g we have that

$$g'(\xi) = (2\beta-3)\xi^{2\beta-4} + \frac{2(\beta-2)C_8}{R_L^{\frac{\beta}{\beta-2}}} \xi^{2\beta-5},$$

hence $g' > 0$ in $(0, \infty)$ and so the function g is strictly increasing in $(0, \infty)$. Since g is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(\xi_*, 0)$. So

$$\xi_*^{2\beta-3} \leq \frac{C_9}{R_L^{\frac{\beta}{\beta-2}}} \quad \text{or} \quad \xi_* \leq C_9^{\frac{1}{2\beta-3}} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}.$$

Furthermore, since the function g is strictly increasing, we get

$$\xi_*^{2\beta-3} + \frac{C_8}{R_L^{\frac{\beta}{\beta-2}}} \left(C_9^{\frac{1}{2\beta-3}} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}} \right)^{2(\beta-2)} - \frac{C_9}{R_L^{\frac{\beta}{\beta-2}}} \geq 0,$$

i.e.,

$$\xi_*^{2\beta-3} \geq \frac{C_9}{R_L^{\frac{\beta}{\beta-2}}} - \frac{C_8 C_9^{\frac{2(\beta-2)}{2\beta-3}}}{R_L^{\frac{\beta(4\beta-7)}{(\beta-2)(2\beta-3)}}},$$

which gives that (notice that $\frac{4\beta-7}{2\beta-3} > 1$ for $\beta > 2$),

$$\xi_*^{2\beta-3} \geq \frac{1}{2} \frac{C_9}{R_L^{\frac{\beta}{\beta-2}}}, \quad \text{for sufficiently large } R_L.$$

So we have that $\xi_* \geq C_{10} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}$, for sufficiently large R_L, which implies that

$$\theta_*^{-\frac{\beta}{2(\beta-1)(\beta-2)}} \geq C_{10} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}, \text{ i.e., } \theta_* \geq C_{11} R_L^{\frac{2(\beta-1)}{2\beta-3}}, \quad \text{for sufficiently large } R_L. \tag{2.19}$$

From (2.17), $t_* = C_6\theta_*^{-\frac{\beta}{\beta-1}}$, whence, in view of (2.19),

$$t_* \geq C_6 C_{11}^{-\frac{\beta}{\beta-1}} R_L^{-\frac{2\beta}{2\beta-3}}, \quad \text{for sufficiently large } R_L. \tag{2.20}$$

Finally, combining (2.19) and (2.20) we have that

$$\theta_* t_* \geq C_{11} R_L^{\frac{2(\beta-1)}{2\beta-3}} C_6 C_{11}^{-\frac{\beta}{\beta-1}} R_L^{-\frac{2\beta}{2\beta-3}} = C_{12} R_L^{-\frac{2}{2\beta-3}}, \quad \text{for sufficiently large } R_L,$$

where C_{12} is a suitable positive constant. Therefore, the following has been proved.

Theorem 2.1. *For large t, the function $u(x,t)$ is analytic in x in a strip of width*

$$\gamma_L \geq k R_L^{-\frac{2}{2\beta-3}},$$

around the real axis. □

Remark 2.1. It is well known for the KS equation (1.1), that for large t, the function $u(x,t)$ is analytic in x in a strip of width $\gamma_L \geq k R_L^{-\frac{2}{5}}$, around the real axis. This arises and from Theorem 2.1 with $\beta = 4$.

2.2. Analyticity of DMKS and HTKS equations

Here we study the analyticity properties of the equations

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u = 0, \tag{2.21}$$

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mu\mathcal{H}[u]_{xxx} = 0, \tag{2.22}$$

for initial data which are periodic with period 2π, where $\nu > 0$, $\mu \geq 0$, $\mathcal{D}$ is a dispersive pseudo-differential operator and $\mathcal{H}$ is the Hilbert transform operator. We prove that the solutions of the above two equations are analytic in a strip around the real axis, and that the width of the strip of analyticity for (2.21), β_ν say, satisfies the bound

$$\beta_\nu \;\geq\; b\,\nu^{\frac{41}{50}},$$

where b is a positive constant, and for (2.22), $\delta_{\nu,\mu}$ say, satisfies the bound

$$\delta_{\nu,\mu} \;\geq\; d\left(\frac{\nu}{\mu}\right)^{41/25},$$

where d is a positive constant. Equation (2.21) is obtained from the following equation given on L-periodic interval

$$u_t + uu_x + u_{xx} + u_{xxxx} + \mathcal{D}u = 0,$$

by the following rescaling (dropping the bars):

$$\bar{t} = \nu t, \quad \bar{x} = \nu^{\frac{1}{2}} x, \quad \bar{u} = \nu^{-\frac{1}{2}} u,$$

where $\nu = (\frac{2\pi}{L})^2$. Equation (2.22) is obtained from the following equation given on L-periodic interval

$$u_t + uu_x + u_{xx} + u_{xxxx} + \gamma\mathcal{H}[u]_{xxx} = 0,$$

where $\gamma \geq 0$, by the following rescaling (dropping the bars):

$$\bar{t} = \nu t, \quad \bar{x} = \nu^{\frac{1}{2}} x, \quad \bar{u} = \nu^{-\frac{1}{2}} u,$$

where $\nu = (\frac{2\pi}{L})^2$ and $\mu = \frac{2\pi}{L}\gamma$.

2.3. Analyticity of solutions for the DMKS equation

Let u be an 2π-periodic in the first variable, function, such that $u(\cdot,t)$ has vanishing mean value, for all $t \geq 0$, and satisfies the equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{2.23}$$

where ν is a positive constant and $\mathcal{D}$ is a dispersive pseudo-differential operator. For a positive constant α and the operator A introduced in (2.3), we define the function v by

$$v(x,t) := (e^{\alpha t A}u)(x,t). \tag{2.24}$$

Then, (2.23) takes the form

$$(e^{-\alpha t A})v_t - \alpha e^{-\alpha t A}Av + e^{-\alpha t A}(v_{xx} + \nu v_{xxxx}) + uu_x + \mathcal{D}u = 0. \tag{2.25}$$

Taking in (2.25) the L^2 inner product with $e^{\alpha t A}v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(\cdot,t)\|^2 - \alpha(Av,v) - \|v_x\|^2 + \nu\|v_{xx}\|^2 + (\mathcal{D}u, e^{\alpha t A}v) + (uu_x, e^{\alpha t A}v) = 0. \tag{2.26}$$

Now, combining (2.26) with (2.6), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \alpha\|A^{\frac{1}{2}}v\|^2 + \|Av\|^2 - \nu\|A^2v\|^2 + C\sqrt{\alpha t}\,\|v\|\,\|Av\|^2. \tag{2.27}$$

Now, in view of Lemma 2.2, we get, respectively, with $\alpha = 2, \beta = 4$, and $\alpha = 1, \beta = 4$ that

$$\|Av\|^2 \leq \|v\|\,\|A^2v\| \ \text{ and} \tag{2.28}$$

$$\|A^{\frac{1}{2}}v\|^2 \leq \|v\|^{\frac{3}{2}}\,\|A^2v\|^{\frac{1}{2}}. \tag{2.29}$$

Combination of (2.27), (2.28) and (2.29) provides that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \alpha\|v\|^{\frac{3}{2}}\,\|A^2v\|^{\frac{1}{2}} + \|v\|\,\|A^2v\| - \nu\|A^2v\|^2 + C\sqrt{\alpha t}\,\|v\|^2\,\|A^2v\|.$$

Using here Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq{}& \frac{\left(\frac{\alpha}{\varepsilon_1}\|v\|^{\frac{3}{2}}\right)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{\left(\varepsilon_1\|A^2v\|^{\frac{1}{2}}\right)^4}{4} + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2v\|^2 - \nu\|A^2v\|^2\\ &+ \frac{C^2}{2\varepsilon_3^2}\alpha t\,\|v\|^4 + \frac{\varepsilon_3^2}{2}\|A^2v\|^2,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq{}& \frac{3\alpha^{\frac{4}{3}}}{4\varepsilon_1^{\frac{4}{3}}}\|v\|^2 + \frac{\varepsilon_1^4}{4}\|A^2v\|^2 + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2v\|^2 - \nu\|A^2v\|^2 + \frac{C^2}{2\varepsilon_3^2}\alpha t\,\|v\|^4\\ &+ \frac{\varepsilon_3^2}{2}\|A^2v\|^2,\end{aligned}$$

whence, choosing $\varepsilon_1 = \nu^{\frac{1}{4}}, \varepsilon_2 = \frac{1}{\nu}, \varepsilon_3 = \nu^{\frac{1}{2}}$ we get

$$\frac{d}{dt}\|v\|^2 \leq \Big(\frac{2}{\nu} + \frac{3\alpha^{\frac{4}{3}}}{2\nu^{\frac{1}{3}}}\Big)\|v\|^2 + \frac{C^2}{\nu}\alpha t\,\|v\|^4. \tag{2.30}$$

With $\Phi(t) := \|v(\cdot,t)\|^2$, we write (2.30) in the form

$$\Phi'(t) \le \Big(\frac{C_1}{\nu} + \frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)\Phi(t) + \frac{C_3}{\nu}\alpha t(\Phi(t))^2. \tag{2.31}$$

Assume now that $\Phi(0) \le R_\nu^2$. As long as $\Phi(t) \le 4R_\nu^2$ holds, relation (2.31) implies

$$\Phi'(t) \le \Big(\frac{C_1}{\nu} + \frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + 4\frac{C_3}{\nu}R_\nu^2\alpha t\Big)\Phi(t),$$

whence

$$\Phi(t) \le \Phi(0)\exp\left[\Big(\frac{C_1}{\nu} + \frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)t + 2\frac{C_3}{\nu}R_\nu^2\alpha t^2\right].$$

As long as

$$\Big(\frac{C_1}{\nu} + \frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)t + 2\frac{C_3}{\nu}R_\nu^2\alpha t^2 \le \log 4,$$

we have $\Phi(t) \le 4R_\nu^2$. This holds for $t \le t_\nu$ which is the positive root of the quadratic

$$2\frac{C_3}{\nu}R_\nu^2\alpha t^2 + \Big(\frac{C_1}{\nu} + \frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)t - \log 4 \;=\; 0.$$

Since $R_\nu = c_0\nu^{-\frac{21}{20}}$ (see[4]), where c_0 is a constant, the last quadratic takes the form

$$2C_4\nu^{-\frac{31}{10}}\alpha t^2 + C_1\nu^{-1}t + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4 = 0.$$

The objective is to maximize the product αt, for small ν, in order to optimize the width of the strip of analyticity. Now, the technique developed in Section 2 to maximize the product θt, for large R_L, applies, with minor modifications, and in this case; and we prove the following.

Theorem 2.2. *For large t, the function $u(x,t)$ is analytic in x in a strip of width*

$$\beta_\nu \ge b\nu^{\frac{41}{50}},$$

around the real axis.

2.4. Analyticity of solutions for the HTKS equation

Let u be an $2\pi-$periodic in the first variable, function, such that $u(\cdot,t)$ has vanishing mean value, for all $t \ge 0$, and satisfies the equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mu\mathcal{H}[u]_{xxx} \;=\; 0, \quad x \in \mathbb{R},\ t \ge 0, \tag{2.32}$$

where ν is a positive constant, μ is a non negative constant and $\mathcal{H}$ is the Hilbert transform operator defined by

$$\mathcal{H}[f](x) = \frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{f(\xi)}{x-\xi}d\xi,$$

where the integral is understood in the sense of a Cauchy principal value.

Using the definition of the function v, which given by (2.24), equation (2.32) takes the form

$$(e^{-\alpha tA})v_t - \alpha e^{-\alpha tA}Av + e^{-\alpha tA}(v_{xx} + \nu v_{xxxx} + \mu\mathcal{H}[v]_{xxx}) + uu_x = 0. \tag{2.33}$$

Taking in (2.33) the L^2 inner product with $e^{\alpha tA}v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(\cdot,t)\|^2 - \alpha(Av,v) - \|v_x\|^2 + \nu\|v_{xx}\|^2 + \mu(\mathcal{H}[v]_{xxx},v) + (uu_x, e^{\alpha tA}v) = 0. \tag{2.34}$$

Now, combining (2.34) with (2.6), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \alpha\|A^{\frac{1}{2}}v\|^2 + \|Av\|^2 - \nu\|A^2v\|^2 + \mu\|A^{\frac{3}{2}}v\|^2 + C\sqrt{\alpha t}\,\|v\|\,\|Av\|^2. \tag{2.35}$$

Combining (2.28), (2.29) and

$$\|A^{\frac{3}{2}}v\|^2 \le \|v\|^{\frac{1}{2}}\,\|A^2v\|^{\frac{3}{2}},$$

which arises from Lemma 2.2 with $\alpha = 3$ and $\beta = 4$, (2.35) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 &\le \alpha\|v\|^{\frac{3}{2}}\,\|A^2v\|^{\frac{1}{2}} + \|v\|\,\|A^2v\| - \nu\|A^2v\|^2 + \mu\|v\|^{\frac{1}{2}}\,\|A^2v\|^{\frac{3}{2}}\\ &\quad + C\sqrt{\alpha t}\,\|v\|^2\,\|A^2v\|.\end{aligned}$$

Using here Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \le{}& \frac{\left(\frac{\alpha}{\varepsilon_1}\|v\|^{\frac{3}{2}}\right)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{\left(\varepsilon_1\|A^2v\|^{\frac{1}{2}}\right)^4}{4} + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2v\|^2 - \nu\|A^2v\|^2\\ &+ \frac{\left(\frac{\mu}{\varepsilon_3}\|v\|^{\frac{1}{2}}\right)^4}{4} + \frac{\left(\varepsilon_3\|A^2v\|^{\frac{3}{2}}\right)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{C^2}{2\varepsilon_4^2}\alpha t\,\|v\|^4 + \frac{\varepsilon_4^2}{2}\|A^2v\|^2,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \le{}& \frac{3\alpha^{\frac{4}{3}}}{4\varepsilon_1^{\frac{4}{3}}}\|v\|^2 + \frac{\varepsilon_1^4}{4}\|A^2v\|^2 + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2v\|^2 - \nu\|A^2v\|^2\\ &+ \frac{\mu^4}{4\varepsilon_3^4}\|v\|^2 + \frac{3\varepsilon_3^{\frac{4}{3}}}{4}\|A^2v\|^2 + \frac{C^2}{2\varepsilon_4^2}\alpha t\,\|v\|^4 + \frac{\varepsilon_4^2}{2}\|A^2v\|^2,\end{aligned}$$

whence, choosing $\varepsilon_1 = \frac{\nu^{\frac{1}{4}}}{\sqrt{2}}$, $\varepsilon_2 = \frac{2}{\nu}$, $\varepsilon_3 = \nu^{\frac{3}{4}}$, $\varepsilon_4 = \frac{\nu^{\frac{1}{2}}}{\sqrt{8}}$ we get

$$\frac{d}{dt}\|v\|^2 \le \Big(\frac{4}{\nu} + \frac{3\alpha^{\frac{4}{3}}}{2^{\frac{1}{3}}\nu^{\frac{1}{3}}} + \frac{\mu^4}{2\nu^3}\Big)\|v\|^2 + 8\frac{C^2}{\nu}\alpha t\,\|v\|^4. \tag{2.36}$$

With $\Phi(t) := \|v(\cdot,t)\|^2$, we write (2.36) in the form

$$\Phi'(t) \le \Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)\Phi(t) + \frac{D_4}{\nu}\alpha t(\Phi(t))^2. \tag{2.37}$$

Assume now that $\Phi(0) \le R^2_{\nu,\mu}$. As long as $\Phi(t) \le 4R^2_{\nu,\mu}$ holds, relation (2.37) implies

$$\Phi'(t) \le \Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3} + 4\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t\Big)\Phi(t),$$

whence

$$\Phi(t) \le \Phi(0)\exp\Big[\Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)t + 2\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t^2\Big].$$

As long as

$$\Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)t + 2\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t^2 \le \log 4,$$

we obviously have $\Phi(t) \le 4R^2_{\nu,\mu}$. This holds for $t \le t_{\nu,\mu}$ which is the positive root of the quadratic

$$2\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t^2 + \Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)t - \log 4 = 0.$$

Since $R_{\nu,\mu} = d_0 \nu^{-\frac{31}{10}} \mu^{\frac{41}{10}}$ (see [25]), where d_0 is a constant, the last quadratic takes the form

$$D_5 \nu^{-\frac{36}{5}} \mu^{\frac{41}{5}} \alpha t^2 + D_1 \nu^{-1} t + D_2 \nu^{-\frac{1}{3}} \alpha^{\frac{4}{3}} t + D_3 \nu^{-3} \mu^4 t - \log 4 = 0.$$

The objective is to maximize the product αt, for small ν and large μ, in order to optimize the width of the strip of analyticity. Now, the technique developed in Section 2 to maximize the product θt, for large R_L, applies, with minor modifications, and in this case; and we prove the following.

Theorem 2.3. *For large t, the function $u(x,t)$ is analytic in x in a strip of width*

$$\delta_{\nu,\mu} \geq d \left(\frac{\nu}{\mu}\right)^{\frac{41}{25}},$$

around the real axis.

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